Resilient Event-Triggered Consensus with Exponential Convergence in Multi-agent Systems

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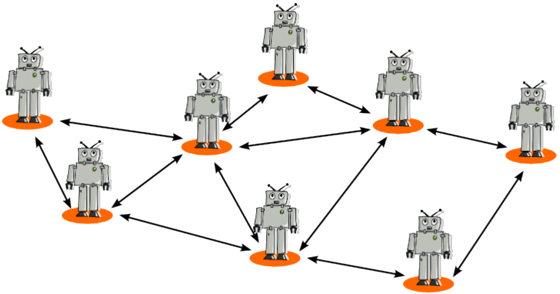
Multi-agent Systems



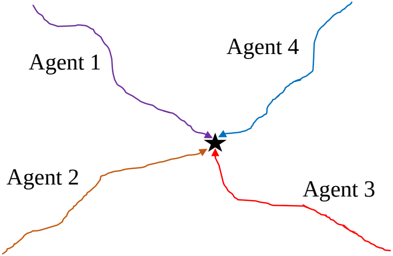
A multi-agent system consists of multiple agents that interact to achieve a cooperative objective.

An agent can represent a moving vehicle, a sensor node, an electric bus, etc.

Cooperative Objectives: Formation, Consensus, Containment, Rendezvous, ...

Consensus

Consensus Control:



★ The agreement value

Consensus: To reach an agreement upon a common value.

Agent Dynamics

General Linear Agent Model :

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{B}\boldsymbol{u}_i(t), \quad 1 \leq i \leq N \tag{1}$$

- $\boldsymbol{x}_i(t) \in \mathbb{R}^n$: The state of agent i at time instant t.
- $\boldsymbol{A} \in \mathbb{R}^{n \times n}$: System matrix (known and constant).
- $\boldsymbol{B} \in \mathbb{R}^{n \times m}$: Input matrix (known and constant).
- $\boldsymbol{u}_i(t) \in \mathbb{R}^{m \times n}$: The proposed distributed control law.
- N : Number of agents in the network.

Consensus

Consensus Definition:

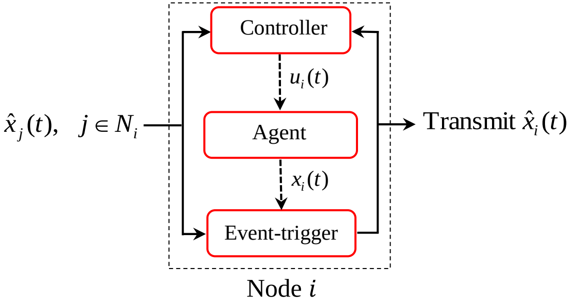
For any initial condition $\boldsymbol{x}_i(0)$, the consensus problem for (1) is said to be solved in the global sense iff **:**

- $\lim_{t\to\infty} \| \boldsymbol{x}_i(t) - \boldsymbol{x}_j(t) \| = 0, \quad (1 \le i, j \le N),$

Key components in reaching consensus:

- The distributed control law that generates $\boldsymbol{u}_i(t)$.
- Information exchange between the neighbouring agents.

Event-triggered Consensus



1. $\mathbf{x}_i(t)$: The state of agent i.
2. $\hat{\mathbf{x}}_i(t)$: The last transmitted state of agent i up to time t which is determined by the event-triggering function.

Motivations

Motivations:

- Transmission saving for consensus in multi-agent systems with bandwidth constrained environments.
- Improve consensus performance in terms of convergence rate and energy consumption.
- Propose a co-design approach to compute control and transmission parameters.

Objectives

Objectives:

- Achieve event-triggered consensus with a desired exponential rate of convergence (as opposed to asymptotic rate).
- Compute optimal consensus parameters with respect to an objective function.
- Guarantee a level of resilience to norm-bounded uncertainties in design parameters.

Assumptions

Assumptions:

- The pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable.
- All agent states are available through measurement or observes.
- Transmission scheme is event-triggered; control input is continuously updated.
- The network is directed and strongly connected.
- Network connectivity information is known for the parameter design stage.
- Uncertainties are norm-bounded and predetermined.

Disagreement vector

- Event-based disagreement vector :

$\boldsymbol{q}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \, (e^{\boldsymbol{A}(t-t^i_{k_i})} \boldsymbol{x}_i(t^i_{k_i}) - e^{\boldsymbol{A}(t-t^j_{k_j})} \boldsymbol{x}_j(t^j_{k_j})).$

a_{ij}: The weight for channel link between agent i and j.

$t^i_{k_i}$: The k_i-th event time for agent i.

- Measurement error : $\boldsymbol{e}_i(t) = e^{\boldsymbol{A}(t-t^i_{k_i})} \boldsymbol{x}_i(t^i_{k_i}) - \boldsymbol{x}_i(t).$

The measurement error is an open-loop estimation of $\boldsymbol{x}_i(t)$ in $t^i_{k_i} \leq t < t^i_{k_i+1}$.

Transmission Scheme

- Event-triggering function :

Given an event time $t^i_{k_i}$, the next event for agent i is triggered at $t = t^i_{k_i+1}$, where

$$t^i_{k_i+1} = \inf\{ t > t^i_{k_i} \mid \|\boldsymbol{e}_i(t)\| - \phi_i \|\boldsymbol{q}_i(t)\| \geq 0 \}, \tag{2}$$

$\phi_i > 0$: The uncertain threshold in the form of $\phi_i = \phi + \delta_{\phi_i}(t)$.

ϕ : Transmission threshold to be designed.

$\delta_{\phi_i}(t)$: Uncertainty in the designed transmission threshold.

$$\| \delta_{\phi_i}(t) \| \leq \alpha \phi, \text{ where } 0 < \alpha < 1.$$

Control law

- The proposed control law :

$$\boldsymbol{u}_i(t) = \bar{\boldsymbol{K}}_i \boldsymbol{q}_i(t), \tag{3}$$

$\bar{\boldsymbol{K}}_i = \boldsymbol{K}_i + \boldsymbol{\Delta}_{\boldsymbol{K}_i}(t)$.

$\boldsymbol{K}_i$: Control gain to be designed.

$\boldsymbol{\Delta}_{\boldsymbol{K}_i}(t)$: Control gain uncertainty.

$$\|\boldsymbol{\Delta}_{\boldsymbol{K}_i}(t)\| \leq \delta_i.$$

Design Objective

Question:

How to design optimal[1] values for transmission threshold ϕ and control gain $\boldsymbol{K}_i$ that guarantee an exponential rate of consensus with norm-bounded design parameter uncertainties?

[1] maximize ϕ to minimize events, and minimize $\boldsymbol{K}_i$ to minimize control force

Preliminary steps prior to optimization

- Augmented closed-loop system :

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A}_{[N]} + \boldsymbol{B}\bar{\boldsymbol{K}}\boldsymbol{L})\boldsymbol{x}(t) + \boldsymbol{B}\bar{\boldsymbol{K}}\boldsymbol{L}\boldsymbol{e}(t), \quad (4)$$

$\boldsymbol{L}$: Laplacian matrix

  - Convert the consensus problem into an equivalent stability problem → Lyapunov stability method.

- System transformation :

$$\boldsymbol{x}_{\mathrm{r}}(t) = \hat{\boldsymbol{L}}\boldsymbol{x}(t). \quad (5)$$

$\hat{\boldsymbol{L}} \in \mathbb{R}^{(N-1)\times N}$ is obtained by removing an arbitrary row of $\boldsymbol{L}$ to eliminate its redundancy.

Preliminary steps prior to optimization

- Transformed (reduced-order) system :

$$\dot{\boldsymbol{x}}_{\mathrm{r}}(t) = (\,\boldsymbol{A}_{[N-1]} + \mathcal{A} + \Delta_{\mathcal{A}}\,)\,\boldsymbol{x}_{\mathrm{r}}(t) + (\,\mathcal{A} + \Delta_{\mathcal{A}}\,)\,\boldsymbol{e}_{\mathrm{r}}(t), \qquad (6)$$

$$\boldsymbol{A}_{[N-1]} = \boldsymbol{I}_{N-1} \otimes \boldsymbol{A}, \qquad \mathcal{A} = \hat{\boldsymbol{L}}\boldsymbol{B}\boldsymbol{K}\mathbb{L},$$

$$\Delta_{\mathcal{A}} = \hat{\boldsymbol{L}}\boldsymbol{B}\boldsymbol{\Delta}_{\boldsymbol{K}}(t)\mathbb{L}, \qquad \boldsymbol{e}_{\mathrm{r}}(t) = \hat{\boldsymbol{L}}\boldsymbol{e}(t),$$

$$\mathbb{L} = \boldsymbol{L}\hat{\boldsymbol{L}}^{\dagger}.$$

Stability of system (6) is equivalent to Consensus in system (4)

Preliminary steps prior to optimization

The event-triggering condition needs to be expressed by $\boldsymbol{x}_{\text{r}}(t)$ and $\boldsymbol{e}_{\text{r}}(t)$ $\rightarrow$ Co-design approach.

The following inequality can be obtained from the event-triggering condition (2) :

$$\boldsymbol{e}_{\text{r}}^{T}(t)\boldsymbol{e}_{\text{r}}(t) \leq \left(\boldsymbol{e}_{\text{r}}(t)+\boldsymbol{x}_{\text{r}}(t)\right)^{T}\boldsymbol{M}^{T}\phi^{2}\boldsymbol{M}\left(\boldsymbol{e}_{\text{r}}(t)+\boldsymbol{x}_{\text{r}}(t)\right). \qquad (7)$$

Inequality (7) is the sufficient event-triggering condition in the optimization stage based on which parameter ϕ can be obtained.

Preliminary steps prior to optimization

Definition

Exponential Stability:
Given convergence rate $\zeta > 0$, system (6) is ζ-exponentially stable if there exists a positive scalar η such that $\| \boldsymbol{x}_r(t) \| \leq \eta\, e^{-\zeta t} \| \boldsymbol{x}_r(0) \|$, $t \geq 0$ for any initial condition $\boldsymbol{x}_r(0)$.

Exponential Lyapunov stability for system (6):

$$\dot{V}(t) + 2\zeta V(t) < 0, \tag{8}$$

with $V(t) = \boldsymbol{x}_r^T(t)\boldsymbol{P}\boldsymbol{x}_r(t)$.

Inequality (8) leads to:

$\lambda_{\min}(\boldsymbol{P})\|\boldsymbol{x}_r(t)\|^2 \leq V(t) < V(0)e^{-2\zeta t} \leq \lambda_{\max}(\boldsymbol{P})e^{-2\zeta t}\|\boldsymbol{x}_r(0)\|^2,$

Compute optimal consensus parameters

The exponential Lyapunov inequality (8) along with the event-triggering condition (7) lead to following matrix inequality

$$\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{\Pi}_1 & \boldsymbol{\Pi}_2 \\ * & \boldsymbol{\Pi}_3 \end{bmatrix} < 0, \tag{9}$$

where

$$\boldsymbol{\Pi}_1 = \begin{bmatrix} \boldsymbol{\pi}_{11} & \boldsymbol{\Xi}\mathbb{L} \\ * & -\tau_1 \boldsymbol{I} + \tau_3 \delta^2 \mathbb{L}^T \mathbb{L} \end{bmatrix}, \qquad \boldsymbol{\Pi}_2 = \begin{bmatrix} \bar{\boldsymbol{P}}\hat{\boldsymbol{L}}\boldsymbol{B} & \bar{\boldsymbol{P}}\hat{\boldsymbol{L}}\boldsymbol{B} & \boldsymbol{M}^T \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}^T \end{bmatrix},$$
$$\boldsymbol{\Pi}_3 = \mathrm{diag}(-\tau_2 \boldsymbol{I}, -\tau_3 \boldsymbol{I}, -\mu \boldsymbol{I}), \qquad \bar{\boldsymbol{P}} = \boldsymbol{I}_{N-1} \otimes \boldsymbol{P}, \tag{10}$$

with

$$\boldsymbol{\pi}_{11} = \boldsymbol{A}_{[N-1]}^T \bar{\boldsymbol{P}} + \bar{\boldsymbol{P}} \boldsymbol{A}_{[N-1]} + \boldsymbol{\Xi}\mathbb{L} + \mathbb{L}^T \boldsymbol{\Xi}^T + 2\zeta \bar{\boldsymbol{P}} + \tau_2 \delta^2 \mathbb{L}^T \mathbb{L},$$
$$\boldsymbol{\Xi} = (\hat{\boldsymbol{L}} \otimes \boldsymbol{1}_n \boldsymbol{1}_n^T) \circ \Big(\boldsymbol{1}_{N-1} \otimes [\, \boldsymbol{\Theta}_1 , \dots , \boldsymbol{\Theta}_N \,] \Big). \tag{11}$$

- $\boldsymbol{\Theta}_i$ $(1 \le i \le N)$, μ, τ_j $(1 \le j \le 3)$, and $\boldsymbol{P}$ are decision variables;

Compute optimal consensus parameters

Consensus parameters are obtained from **:**

$$\phi = \sqrt{{\tau_1}^{-1}\mu^{-1}}, \text{ and } \boldsymbol{K}_i = \boldsymbol{B}^{\dagger}\boldsymbol{P}^{-1}\boldsymbol{\Theta}_i, \quad (1 \leq i \leq N) \qquad (12)$$

To enlarge ϕ and restrict $\boldsymbol{K}_i$ as much as possible, we minimize decision variables involved in obtaining them, i.e., τ_1, μ, $\boldsymbol{P}$, and $\boldsymbol{\Theta}_i$ by introducing bounding scalars ω_c $(1 \leq c \leq N+3)$, i.e.,

$$\min_{\Theta_i,\mu,\tau_j,\boldsymbol{P},\omega_c} \quad J = \overbrace{nN(\omega_1+\omega_2)}^{\text{To enlarge } \phi} + \overbrace{\text{Tr}(\boldsymbol{\Omega}_P)+\text{Tr}(\boldsymbol{\Omega}_\Theta)}^{\text{To restrict } \boldsymbol{K}_i}$$

subject to

$$\begin{bmatrix} -\omega_1 & \tau_1 \\ * & -1 \end{bmatrix} < 0, \begin{bmatrix} -\omega_2 & \mu \\ * & -1 \end{bmatrix} < 0, \begin{bmatrix} \boldsymbol{\Omega}_P & \boldsymbol{I} \\ * & \boldsymbol{I}_N \otimes \boldsymbol{P} \end{bmatrix} > 0, \begin{bmatrix} -\boldsymbol{\Omega}_\Theta & \boldsymbol{\Theta}^T \\ * & -\boldsymbol{I} \end{bmatrix} < 0,$$

with $\quad \boldsymbol{\Omega}_P = \omega_3 \boldsymbol{I}_{Nn}, \qquad \boldsymbol{\Omega}_\Theta = \text{diag}\left(\omega_4 \boldsymbol{I}_n, \ldots, \omega_{N+3}\boldsymbol{I}_n\right),$

Compute optimal consensus parameters

Consensus parameters are then guaranteed to be bounded by the following terms :

$$\phi \geq (\omega_1 \omega_2)^{\frac{-1}{4}},$$
$$\boldsymbol{K}_i \boldsymbol{K}_i^T \leq \omega_{i+3}\, \omega_3^2 \boldsymbol{B}^\dagger {\boldsymbol{B}^\dagger}^T, \ (1 \leq i \leq N). \quad (13)$$

Compute optimal consensus parameters

In summary : Solve the following optimization problem for desired convergence rate ζ and resilient level δ and α:

$$\min_{\Theta_i,\mu,\tau_j,\boldsymbol{P},\omega_c} \quad J = nN(\omega_1+\omega_2) + \mathrm{Tr}(\boldsymbol{\Omega}_P) + \mathrm{Tr}(\boldsymbol{\Omega}_\Theta) \qquad (14)$$

subject to

$$\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{\Pi}_1 & \boldsymbol{\Pi}_2 \\ * & \boldsymbol{\Pi}_3 \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\omega_1 & \tau_1 \\ * & -1 \end{bmatrix} < 0, \begin{bmatrix} -\omega_2 & \mu \\ * & -1 \end{bmatrix} < 0, \begin{bmatrix} \boldsymbol{\Omega}_P & \boldsymbol{I} \\ * & \boldsymbol{I}_N \otimes \boldsymbol{P} \end{bmatrix} > 0, \begin{bmatrix} -\boldsymbol{\Omega}_\Theta & \boldsymbol{\Theta}^T \\ * & -\boldsymbol{I} \end{bmatrix} < 0,$$

Once the optimization problem (14) is solved, compute consensus parameters :

$$\phi = \sqrt{{\tau_1}^{-1}\mu^{-1}}, \quad \text{and} \quad \boldsymbol{K}_i = \boldsymbol{B}^\dagger \boldsymbol{P}^{-1}\boldsymbol{\Theta}_i, \quad (1 \le i \le N) \qquad (15)$$

Zeno-behaivoir Exclusion

There must always be a finite number of events within a given finite time interval, Otherwise → Zeno-behaviour

The time interval between any two consecutive events is lower-bounded by the following term → Zeno-behaviour is excluded

$$t^i_{k_i+1} - t^i_{k_i} \geq \frac{1}{2||\boldsymbol{A}||} \ln\left(1 + \frac{2\phi_i||\boldsymbol{A}||}{||\boldsymbol{B}\bar{\boldsymbol{K}}_i||}\right) \tag{16}$$

Experimental Results

- A network of six robot agents :

$$\begin{aligned} \dot{\boldsymbol{x}}_i(t) &= v_i(t)\cos(\theta_i(t)), \\ \dot{\boldsymbol{y}}_i(t) &= v_i(t)\sin(\theta_i(t)), \\ \dot{\theta}_i(t) &= \omega_i(t), \qquad (1 \le i \le 6), \end{aligned} \tag{17}$$

$\boldsymbol{x}_i(t) \in \mathbb{R}$, and $\boldsymbol{y}_i(t) \in \mathbb{R}$: Cartesian coordinates of the mass center for robot i.

$v_i(t) \in \mathbb{R}$: Linear velocity.

$\theta_i(t) \in \mathbb{R}$: Heading angle.

$\omega_i(t) \in \mathbb{R}$: Angular velocity.

- Feedback Linearizion

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{18}$$

Experimental Results

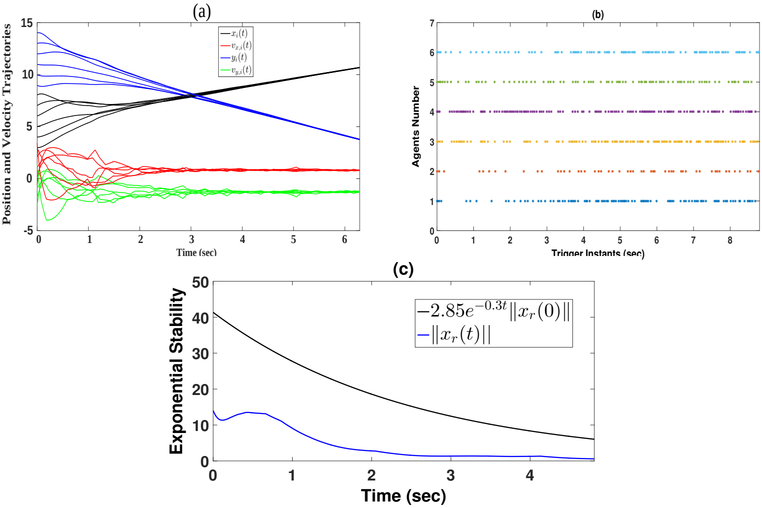
- Laplacian Matrix :

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{19}$$

- To solve the event-triggered consensus using optimization (14), we consider $\zeta = 0.4$, $\delta_i = 0.02$, and $\alpha = 0.1$.
- Solve the optimization (14) to compute $\boldsymbol{K}_i$ and ϕ

$$\begin{aligned} \boldsymbol{K}_1 &= -\boldsymbol{I}_2 \otimes [\,3.9950, 3.4909\,],\ \boldsymbol{K}_2 = -\boldsymbol{I}_2 \otimes [\,1.7339, 1.4982\,], \\ \boldsymbol{K}_3 &= -\boldsymbol{I}_2 \otimes [\,3.6374, 3.1703\,],\ \boldsymbol{K}_4 = -\boldsymbol{I}_2 \otimes [\,3.6468, 3.2010\,], \\ \boldsymbol{K}_5 &= -\boldsymbol{I}_2 \otimes [\,2.5755, 2.2525\,],\ \boldsymbol{K}_6 = -\boldsymbol{I}_2 \otimes [\,4.4245, 3.8699\,], \\ &\text{and } \ \phi = 0.1520. \end{aligned} \tag{20}$$

Experimental Results



Experimental Results

How different values for convergence rate ζ affect the consensus?

$\overline{\text{TI}}$ **:** Time to reach consensus.
$\overline{\text{AT}}$ **:** Average number of transmission per agent.

Table: Consensus performance; varying ζ with fixed $\{\delta = 0.02, \alpha = 0.1\}$

convergence rate ζ	$\overline{\text{TI}}$	$\overline{\text{AT}}$	J
0.20	925	71.33	149.0563
0.30	579	51.33	151.0456
0.40	879	95.33	153.2891
0.50	631	68.83	154.7435

Increasing $\zeta \rightarrow \overline{\text{TI}}$ constantly gets reduced.

Faster convergence rate $\rightarrow$ higher minimized objective function $J \rightarrow$ larger control gains and a smaller transmission threshold.

Experimental Results

How different values for uncertainty δ affect the consensus?

Table: Consenssus performance, varying δ with fixed $\{\zeta = 0.4, \alpha = 0.1\}$

control gain uncertainty δ	$\overline{\text{TI}}$	$\overline{\text{AT}}$	J
0.01	818	84.5	153.0709
0.03	653	65.16	153.5047
0.05	576	58.16	153.9978
0.07	617	66.17	154.3972
0.08	642	64.33	154.5632

Increasing $\delta \rightarrow$ higher value for the objective function J (smaller ϕ and/or larger $\boldsymbol{K}_i$'s).

Conclusion

1. For a desired rate of convergence, the event-triggered consensus is reached with resilient parameters;
2. Using convex optimization, the transmission threshold ϕ is enlarged (to reduce transmission events) and control gain $\boldsymbol{K}_i$ is restricted (to restrict the control force);
3. As convergence rate ζ is increased, the consensus time constantly gets reduced until the optimization problem becomes infeasible.

Question?

Thank You